Non-local symmetries via Darboux transformations

1997 J. Phys. A: Math. Gen. 30 L95

(http://iopscience.iop.org/0305-4470/30/5/004)

LETTER TO THE EDITOR

Non-local symmetries via Darboux transformations

Sen-Yue Lou†‡ and Xing-Biao Hu†§

† CCAST (World Laboratory), PO Box 8730, Beijing 100080, People's Republic of China
‡ Institute of Modern Physics, Ningbo Normal College, Ningbo 315211, People's Republic of China
§ State Key Laboratory of Scientific and Engineering Computing, Institute of Computational Mathematics and Scientific Engineering Computing, Academia Sinica, PO Box 2719, Beijing 100080, People's Republic of China†

Received 12 November 1996

Abstract. Darboux transformations are used to find non-local symmetries of integrable nonlinear models which include the KdV, KP, CDGKS and (2 + 1)-dimensional CDGKS equations.

An infinite number of symmetries and Darboux transformations are two intriguing properties which had been shown to be among the most intrinsic features of soliton systems such as the Korteweg de Vries (KdV), sine-Gordon (SG) and nonlinear Schrodinger (NLS) equations. Roughly speaking, a symmetry group of a system of differential equations is a group which transforms solutions of the system to other solutions. In general, we can derive corresponding symmetries from invariant transformations of a differential equation under consideration. Let us take the KdV equation as an example. The KdV equation is

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1}$$

where the subscripts represent derivatives. x-translation, t-translation and Galilean invariances of (1) lead to symmetries: u_x, u_t and $tu_x + \frac{1}{6}$, respectively. Recently, one of the authors (Lou) re-obtained the non-local symmetry $\sigma = (\phi^2)_x$ from the conformal invariance of the Schwartz form of the KdV equation (1) [1], where ϕ is a spectral function of Lax pair

$$-\phi_{xx} + u\phi = \lambda\phi \tag{2}$$

$$\phi_t = -u_x\phi + (2u + 4\lambda)\phi_x. \tag{3}$$

It should be noted that the non-local symmetry $\sigma = (\phi^2)_x$ is closely connected with the squared eigenfunctions of the Lax operators in the inverse scattering transform. This type of symmetry has been recognized for over 20 years as the solution of the linearized integrable equations, originally by Kaup [2] (this is an extension of the observation for the KdV by Gardner *et al* [3]).

On the other hand, the Darboux transformation (DT) is the most direct and yet elementary approach for the construction of exact solutions (see, e.g., [4–8]). Using the DT

† Mailing address.

0305-4470/97/050095+06$19.50

method, we can obtain new solutions from old solutions. That means the DTs of integrable equations reflect some invariant properties of the equations under consideration.

It is known that to search for non-local symmetries is of considerable interest. The non-local symmetries thus obtained enlarge the class of symmetries and are connected with integrable models. Now a natural problem is how to find non-local symmetries? An effective method for finding non-local symmetries seems to be to find the inverse of the corresponding recursion operators (see [9, 10]). However, to find the inverse of the recursion operators is in itself a difficult problem.

The main purpose of this paper is to show how these kinds of non-local symmetries can be obtained natually by Darboux transformations. We will use the invariant properties of differential equations exhibited by DTs to deduce the non-local symmetries of the KdV, KP, CDGKS and $(2+1)$-dimensional CDGKS equations.

Let us first consider the KdV equation (1). In this case, we know that:

Proposition 1 [5, 7, 8]. Let u be a solution of the KdV equation (1), where ϕ satisfies (2), (3). Then $\bar{u} = u - 2\partial^2 \ln \phi/\partial x^2$ is a solution of (1).

Now using the DT above, we have

Proposition 2. $\sigma = \left(\tilde{\psi}/\psi\right)_{xx}$ is a symmetry of the KdV equation (1), where $\tilde{\psi}(x,t)$, $\psi(x,t)$ satisfy the following equations:

$$\psi_{xx} - (u + 2(\ln\,\psi)_{xx})\,\psi = 0 \tag{4}$$

$$\psi_t + (u_x + 2(\ln\,\psi)_{xxx})\,\psi - 2\,(u + 2(\ln\,\psi)_{xx})\,\psi_x = 0 \tag{5}$$

$$-\tilde{\psi}_{xx} + (u + 2(\ln\,\psi)_{xx})\,\tilde{\psi} = \psi \tag{6}$$

$$\tilde{\psi}_t + (u_x + 2(\ln\,\psi)_{xxx})\,\tilde{\psi} - 2\,(u + 2(\ln\,\psi)_{xx})\,\tilde{\psi}_x = 4\psi_x. \tag{7}$$

Proof. Set $U = u - 2\partial^2 \ln \phi(x,t,0)/\partial x^2$. From proposition 1, we know that U is a solution of the KdV equation (1). Now we formally expand $\bar{u}$ in powers of λ. We have

$$\bar{u} = U + \lambda\left[\left(-2\frac{\partial^2}{\partial x^2}\ln\,\phi\right)_{\lambda}\Big|_{\lambda=0}\right] + \mathrm{O}(\lambda^2).$$

Thus $(\partial^2 \ln\,\phi/\partial x^2)_\lambda|_{\lambda=0}$ is a symmetry of the KdV equation with respect to U. Finally, substituting $u = U + 2\partial^2 \ln \phi(x,t,0)/\partial x^2$ in (2), (3) leads to (4)–(7) with U replaced by u, $\phi(x,t,0)$ by $\psi(x,t)$ and $\phi_\lambda(x,t,0)$ by $\tilde{\psi}(x,t)$. Thus we have completed the proof of proposition 2.

Furthermore, a direct calculation shows that if ψ satisfies (4), (5), then

$$\tilde{\psi} = -\psi\int_{x_0}^{x}\left[\frac{1}{\psi^2}\int_{x_0}^{x}\psi^2\,\mathrm{d}x\right]\mathrm{d}x + A(t)\psi\int_{x_0}^{x}\frac{1}{\psi^2}\,\mathrm{d}x + B(t)\psi$$

is a solution of (6), (7), where

$$A(t) = \int^{t}(2\psi\psi_{xx} - 4\psi_x^2)\big|_{x=x_0}\,\mathrm{d}t \tag{8}$$

$$B(t) = \int^{t}\left(4\frac{\psi_x}{\psi} + 2A(t)\frac{\psi_{xx}}{\psi^3}\right)\bigg|_{x=x_0}\,\mathrm{d}t. \tag{9}$$

Moreover, it can easily be verified that if ψ is a solution of (4), (5), then $\overline{\phi} = 1/\psi$ satisfies (2), (3) with $\lambda = 0$, i.e.

$$-\overline{\phi}_{xx} + u\overline{\phi} = 0$$

$$\overline{\phi}_t = -u_x\overline{\phi} + 2u\overline{\phi}_x.$$

To sum up, we have:

Proposition 3.

$$\sigma = \left[-\overline{\phi}^2 \int_{x_0}^{x} \frac{1}{\overline{\phi}^2}\, \mathrm{d}x + A_1(t)\overline{\phi}^2 + A_2\overline{\phi}^2\right]_x$$

is a non-local symmetry of the KdV equation (1), where $\overline{\phi}(x,t)$ satisfies (2), (3) with $\lambda = 0$, $A_1(t) = \int_{t_0}^{t}(-2\overline{\phi}_{xx}/\overline{\phi}^3)|_{x=x_0}\, \mathrm{d}t$ and A_2 is a constant.

Note that by taking $A_2 \longrightarrow \infty$, and $A_2 = 0$, respectively, we re-obtain the two seed symmetries of the KdV equation [9].

Next we consider the KP equation. The KP equation reads

$$(u_t - 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0 \qquad \sigma^2 = \pm 1. \tag{10}$$

Its Lax pair is

$$\sigma\phi_y + \phi_{xx} - u\phi = \lambda\phi \tag{11}$$

$$\phi_t + 4\phi_{xxx} - 3u_x\phi - 6u\phi_x + 3\sigma(\partial_x^{-1}u_y)\phi = 0. \tag{12}$$

From [7, 8], we know:

Proposition 4. Let u be a solution of the KP equation (10), with ϕ satisfying (11), (12). Then $\overline{u} = u - 2\partial^2 \ln\phi/\partial x^2$ is a solution of (10).

Now using the DT above and in a manner similar to that of proposition 2, we have:

Proposition 5. $\Sigma = \left(\tilde{\psi}/\psi\right)_{xx}$ is a symmetry of the KP equation (10), where $\tilde{\psi}(x,y,t)$, $\psi(x,y,t)$ satisfy the following equations:

$$\sigma\psi_y + \psi_{xx} - (u + 2(\ln\psi)_{xx})\,\psi = 0 \tag{13}$$

$$\psi_t + 4\psi_{xxx} - 3\,(u_x + 2(\ln\psi)_{xxx})\,\psi - 6\,(u + 2(\ln\psi)_{xx})\,\psi_x$$
$$+3\sigma\left(\partial_x^{-1}u_y + 2(\ln\psi)_{xy}\right)\psi = 0 \tag{14}$$

$$\sigma\tilde{\psi}_y + \tilde{\psi}_{xx} - (u + 2(\ln\psi)_{xx})\,\tilde{\psi} = \psi \tag{15}$$

$$\tilde{\psi}_t + 4\tilde{\psi}_{xxx} - 3\,(u_x + 2(\ln\psi)_{xxx})\,\tilde{\psi} - 6\,(u + 2(\ln\psi)_{xx})\,\tilde{\psi}_x$$
$$+\,3\sigma\left(\partial_x^{-1}u_y + 2(\ln\psi)_{xy}\right)\tilde{\psi} = 0. \tag{16}$$

A direct calculation shows that if ψ satisfies (13), (14) and ψ^* satisfies

$$-\sigma\psi^*_y + \psi^*_{xx} - \left(u + 2(\ln\psi^*)_{xx}\right)\psi^* = 0 \tag{17}$$

$$\psi^*_t + 4\psi^*_{xxx} - 3\left(u_x + 2(\ln\psi^*)_{xxx}\right)\psi^* - 6\left(u + 2(\ln\psi^*)_{xx}\right)\psi^*_x$$
$$-\,3\sigma\left(\partial_x^{-1}u_y + 2(\ln\psi^*)_{xy}\right)\psi^* = 0 \tag{18}$$

then

$$\tilde{\psi} = \psi \int_{x_0}^{x} \frac{1}{\psi\psi^*}\, dx \pm \sigma y \psi \mp \psi \int_{y_0}^{y} f(y,t)\, dy + \psi \int^{t} g(t)\, dt$$

is a solution of (15), (16), where

$$f(y,t) = \left[\frac{1}{\psi} \left(\frac{1}{\psi^*} \right)_x - \frac{1}{\psi^*} \left(\frac{1}{\psi} \right)_x \right]_{x=x_0} \tag{19}$$

$$g(t) = \left[6 \frac{u}{\psi\psi^*} + 4 \frac{\psi_{xx}\psi^* + \psi_x\psi_x^* + \psi\psi_{xx}^*}{\psi^2\psi^{*2}} - 8 \frac{\psi_x^2}{\psi^3\psi^*} - 8 \frac{\psi_x^{*2}}{\psi\psi^{*3}} \right]_{x=x_0,\, y=y_0}. \tag{20}$$

Moreover, it can easily be verified that if ψ is a solution of (13), (14) and ψ^* is a solution of (17), (18), then $\phi = 1/\psi^*$ satisfies (11), (12) with $\lambda = 0$ and $\phi^* = 1/\psi$ satisfies

$$-\sigma\phi_y^* + \phi_{xx}^* - u\phi^* = 0 \tag{21}$$

$$\phi_t^* + 4\phi_{xxx}^* - 3u_x\phi^* - 6u\phi_x^* - 3\sigma(\partial_x^{-1}u_y)\phi^* = 0. \tag{22}$$

To sum up, we have:

Proposition 6. $\Sigma = (\phi\phi^*)_x$ is a non-local symmetry of the KP equation (10), where ϕ and ϕ^* satisfy (11), (12) with $\lambda = 0$ and (21), (22).

Remark. This symmetry was obtained in the literature.

We now turn to consider the $(1+1)$-dimensional CDGKS equation. The $(1+1)$-dimensional CDGKS equation under consideration [11] is

$$u_t = u_{xxxxx} + 5u_x u_{xx} + 5uu_{xxx} + 5u^2 u_x. \tag{23}$$

Its Lax pair is

$$\phi_{xxx} + u\phi_x = \lambda\phi \tag{24}$$

$$\phi_t = -9\phi_{xxxxx} - 15u\phi_{xxx} - 15u_x\phi_{xx} - (10u_{xx} + 5u^2)\phi_x. \tag{25}$$

Concerning the $(1+1)$-dimensional CDGKS equation (23), we have:

Proposition 7 [12]. Let u be a solution of the CDGKS equation (23), with ϕ satisfying (24), (25). Then $\overline{u} = u + 6\partial^2 \ln\phi/\partial x^2$ is a solution of (23).

Now using the DT above, we have:

Proposition 8. $\sigma = \left(\tilde{\psi}/\psi\right)_{xx}$ is a symmetry of the CDGKS equation (23), where $\tilde{\psi}(x,t)$, $\psi(x,t)$ satisfy the following equations:

$$\psi_{xxx} + (u - 6(\ln\psi)_{xx})\,\psi_x = 0 \tag{26}$$

$$\psi_t = -9\psi_{xxxxx} - 15\,(u - 6(\ln\psi)_{xx})\,\psi_{xxx} - 15\,(u - 6(\ln\psi)_{xx})_x\,\psi_{xx} - \left[10\,(u - 6(\ln\psi)_{xx})_{xx} + 5\,(u - 6(\ln\psi)_{xx})^2\right]\psi_x \tag{27}$$

$$\tilde{\psi}_{xxx} + (u - 6(\ln\psi)_{xx})\,\tilde{\psi}_x = \psi \tag{28}$$

$$\tilde{\psi}_t = -9\tilde{\psi}_{xxxxx} - 15\,(u - 6(\ln\psi)_{xx})\,\tilde{\psi}_{xxx} - 15\,(u - 6(\ln\psi)_{xx})_x\,\tilde{\psi}_{xx} - \left[10\,(u - 6(\ln\psi)_{xx})_{xx} + 5\,(u - 6(\ln\psi)_{xx})^2\right]\tilde{\psi}_x. \tag{29}$$

Proof. The proof is similar to that of proposition 2.

Finally, we consider the $(2+1)$-dimensional CDGKS or BKP equation. The $(2+1)$-dimensional CDGKS equation under consideration is

$$u_t = u_{xxxxx} + 5u_x u_{xx} + 5uu_{xxx} + 5u^2 u_x + 5u_{xxy} + 5uu_y + 5u_x \partial_x^{-1} u_y - 5\partial_x^{-1} u_{yy}. \tag{30}$$

Its Lax pair is

$$\phi_{xxx} + u\phi_x + \phi_y = \lambda\phi \tag{31}$$

$$\phi_t = -9\phi_{xxxxx} - 15u\phi_{xxx} - 15u_x\phi_{xx} - (10u_{xx} + 5u^2 - 5\partial^{-1}u_y)\phi_x. \tag{32}$$

Concerning the $(2+1)$-dimensional CDGKS equation (30), we have:

Proposition 9 [12]. Let u be a solution of the $(2+1)$-dimensional CDGKS equation (30), with ϕ satisfying (31), (32). Then $\bar{u} = u + 6\partial^2 \ln \phi/\partial x^2$ is a solution of (30).

Now using the DT above, we have:

Proposition 10. $\sigma = \left(\tilde{\psi}/\psi\right)_{xx}$ is a symmetry of the $(2+1)$-dimensional CDGKS equation (30), where $\tilde{\psi}(x, y, t)$, $\psi(x, y, t)$ satisfy the following equations:

$$\psi_{xxx} + (u - 6(\ln\,\psi)_{xx})\,\psi_x + \psi_y = 0 \tag{33}$$

$$\begin{aligned}\psi_t = {} & -9\psi_{xxxxx} - 15\,(u - 6(\ln\,\psi)_{xx})\,\psi_{xxx} - 15\,(u - 6(\ln\,\psi)_{xx})_x\,\psi_{xx}\\ & - \left[10\,(u - 6(\ln\,\psi)_{xx})_{xx} + 5\,(u - 6(\ln\,\psi)_{xx})^2\right.\\ & \left. -\ 5\partial^{-1}\,(u - 6(\ln\,\psi)_{xx})_y\right]\psi_x\end{aligned} \tag{34}$$

$$\tilde{\psi}_{xxx} + (u - 6(\ln\,\psi)_{xx})\,\tilde{\psi}_x + \tilde{\psi}_y = \psi \tag{35}$$

$$\begin{aligned}\tilde{\psi}_t = {} & -9\tilde{\psi}_{xxxxx} - 15\,(u - 6(\ln\,\psi)_{xx})\,\tilde{\psi}_{xxx} - 15\,(u - 6(\ln\,\psi)_{xx})_x\,\tilde{\psi}_{xx}\\ & - \left[10\,(u - 6(\ln\,\psi)_{xx})_{xx} + 5\,(u - 6(\ln\,\psi)_{xx})^2\right.\\ & \left. -\ 5\partial^{-1}\,(u - 6(\ln\,\psi)_{xx})\right]\tilde{\psi}_x.\end{aligned} \tag{36}$$

Furthermore, a direct calculation shows that if ψ satisfies (33), (34), then

$$\tilde{\psi} = y\psi + A$$

is a solution of (35), (36), where A is a constant. Moreover, it can easily be verified that if ψ is a solution of (33), (34), then $\bar{\phi} = 1/\psi$ satisfies (31), (32) with $\lambda = 0$, i.e.

$$\bar{\phi}_{xxx} + u\bar{\phi}_x + \bar{\phi}_y = 0 \tag{37}$$

$$\bar{\phi}_t = -9\bar{\phi}_{xxxxx} - 15u\bar{\phi}_{xxx} - 15u_x\bar{\phi}_{xx} - (10u_{xx} + 5u^2 - 5\partial^{-1}u_y)\bar{\phi}_x. \tag{38}$$

To sum up, we have:

Proposition 11. $\sigma = \bar{\phi}_{xx}$ is a symmetry of the $(2+1)$-dimensional CDGKS equation (30), where $\bar{\phi}(x, y, t)$ satisfies (37), (38).

Remark. This symmetry was obtained by a direct calculation in [13].

In summary, to find non-local symmetries is an interesting but difficult problem. Darboux transformations provide a natural approach for the derivation of non-local symmetries. Since Darboux transformations can be available for most integrable systems, it would be possible to extend the results in the paper to many interesting integrable models and the corresponding non-local symmetries could be derived.

The work was supported by National Natural Science Foundation of China.

References

[1] Lou S-Y 1995 Negative Kadomtsev–Petviashvili hierarchy *Preprint*
[2] Kaup D J 1976 *J. Math. Anal. Appl.* **54** 849–64
[3] Gardner C S, Greene J M, Kruskal M D and Miura R M 1974 *Commun. Pure and Appl. Math.* **27** 97–133
[4] Darboux G 1882 *CR Acad. Sci. Paris* **94** 1456
[5] Wadati M, Sanuki H and Konno K 1975 *Prog. Theor. Phys.* **53** 419
[6] Levi D, Ragnisco O and Sym A 1984 *Nuovo Cimento* B **83** 314
[7] Matveev V B and Salle M A 1990 *Darboux Transformations and Solitons* (Berlin: Springer)
[8] Gu C H *et al* 1990 *Soliton Theory and Its Application* (Hangzhou: Zhejiang Publishing House of Science and Technology)
[9] Lou S-Y 1993 *Phys. Lett.* **302B** 261
Lou S-Y 1993 *Int. J. Mod. Phys.* A (Proc. Suppl.) **3A** 531–4
Lou S-Y 1994 *J. Math. Phys.* **35** 2390
[10] Lou S-Y 1993 *J. Phy. A: Math. Gen.* **26** L789–91
Lou S-Y 1993 *Phys. Lett.* **181A** 13–6
Lou S-Y 1994 *Phys. Lett.* **187A** 239–42
Lou S-Y 1994 *Solitons, Chaos and Fractals* **4** 1961–4
Ruan H-Y and Lou S-Y 1993 *J. Phys. Soc. Japan* **62** 1917–21
Han P and Lou S-Y 1994 *Acta Phys. Sinica* **43** 1042
Lou S-Y and Chen W-Z 1993 *Phys. Lett.* **179A** 271–4
[11] Sawada K and Kotera T 1974 *Prog. Theor. Phys.* **51** 1355–67
Caudrey P J, Dodd R K and Gibbon J D 1976 *Proc. R. Soc.* A **351** 407–22
[12] Geng X G 1989 *Appl. Math.: A J. of Chinese Uni.* **4** 494–7
[13] Lou S-Y 1994 A integrable $2+1$ dimensional Mikhailov–Dodd–Bullough equation *Preprint*